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Study of Chattering Cruise

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That steady-state cruise is not, in general, fuel-range optimal has been shown in studies by Speyer and others. The presently reported investigation makes use of a reduced-order ("energy") model for analysis of "chattering" cruise, an idealization of time-shared operation between two Mach-number/altitude points, proceeding along the general lines of a study by Gilbert and Parsons. The characteristic nonconvexity of the hodograph figure, which leads to chattering, is examined and attempts are made to relate the extent of the effect to aerodynamic and propulsion-system parameters. One such is an analytical attack on a highly simplified vehicle model featuring Mach-number and altitude independence. Another is a computational study of several aircraft modeled with the usual altitude dependence, and with and without Mach dependence. The trends noted are examined further in a look at specific-energy dependence of the hodograph figure's shape. Although substantial improvements in fuel economy appear in low-energy situations, the best improvement found in the cruise-energy range is about 5%.

Introduction

EFFICIENT cruise performance is of increasing importance in many aircraft operations. Models used in analytical studies of cruise include point-mass models, in which lift coefficient and throttle setting are controls, energy models, with altitude and throttle setting as controls, and a model intermediate in order, featuring path angle as a control variable.

In recent years evidence has emerged that classical steady-state cruise is not generally fuel-optimal.[1,3-5,8,10-14] Particularly significant is the work of J. L. Speyer, who has shown[13] that, for point-mass modeling, classical steady-state cruise solutions can include conjugate points and hence are, in general, not minimizing. In the paper[13] Speyer demonstrated a slight decrease in fuel usage by employing oscillating controls. The true character of minimizing arcs, however, has not yet been revealed.

In this paper the simplest model, the energy model, is studied with the idea that oscillations appearing in this model might be related to those in the complete models (see Sec. VI in Ref. 13). The present approach closely parallels that of Gilbert and Parsons,[4] adding results for a spectrum of aircraft and, perhaps, some additional insights.

In the next sections the energy model is discussed and the problem formulated. Subsequently, results are presented, first for a simple analytical model and then for more complex vehicle models which require computer analysis.

Energy Modeling

For point-mass modeling, relevant dynamical equations governing symmetric flight in a vertical plane are

$$\dot{E} = (\eta \bar{T} - D)\, V/W\ ,\quad \dot{W}_f = \eta \bar{Q}\ ,\quad \dot{x} = V\cos\gamma,$$
$$\epsilon \dot{h} = V\sin\gamma\ ,\quad \epsilon m V\dot{\gamma} = L - mg\cos\gamma \qquad (1)$$

Here E is specific energy, $E = h + V^2/2g$, W_f is the weight of fuel consumed, and $\bar{T}$ and $\bar{Q}$ are the maximum thrust and maximum-fuel-flow rate, respectively. Aircraft weight W has been assumed constant. The symbol V should be regarded as shorthand for the quantity $[2g(E-h)]^{1/2}$. The throttle parameter η can be best thought of as a fraction of maximum thrust, but it should be noted that the fuel-flow rate has been assumed also to be proportional to η.

The parameter ϵ has been introduced to motivate a subsequent order reduction in the spirit of an asymptotic expansion.[6] While the present investigation makes use of only an order-zero "outer" variables model, it is appropriate to note the relationship to the more familiar point-mass model. Assuming that altitude and path angle are "fast" variables, one sets ϵ to zero in the system (1). The fourth equation then implies that the path angle is zero and the fifth that lift equals weight. Since the range is montone increasing it is suitable for use as an independent variable. With these changes the reduced system is

$$E' = [\eta \bar{T}(h,M) - D(h,M)]/W$$

and

$$W_f' = \eta \bar{Q}(h,M)/V(h,E) \qquad (2a)$$

The maximum thrust and fuel-flow rate are prescribed functions of altitude and Mach number, while the drag is generally a function of these and the lift coefficient C_L. By virtue of the order reduction, C_L must be chosen so that lift equals weight. Altitude and velocity are related to the energy variable by $E = h + V^2/2g$.

Control variables in the system (2) are the throttle parameter η and the altitude h. These variables must satisfy the constraints

$$0 \le \eta \le 1 \quad \text{and} \quad 0 \le h \le h_{\max}(E) \qquad (2b)$$

where $h_{\max}(E)$ is the maximum altitude for lift equals weight at the given energy (termed the loft ceiling in Ref. 6).

Problem Statement

With the reduced models (2a) and (2b) in hand, consider the following optimal-control problem: Given x_f, E_0, and end conditions

$$E(0) = E(x_f) = E_0, \qquad W_f(0) = 0$$

find $\eta(x)$, $h(x)$ to minimize $W_f(x_f)$.

Bryson et al.[2] consider the Mayer reciprocal of this problem (maximum range on a given amount of fuel) *for the case with the throttle fixed.* They apply the Maximum Principle,[7] which, in this case, leads to a rather simple two-point-boundary-value problem. If one attempts the same

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procedure for the variable-throttle case a fundamental difficulty arises. An optimal control *may not exist!* Whereas there is a least amount of fuel required, the control that achieves this fuel usage may not be smooth enough to qualify as admissible.

The Maximum Principle[7] requires that at each instant one choose the controls, (throttle setting η, and altitude h, in this case) to maximize the dot product $\langle \lambda, f \rangle$, where λ satisfies a linear system which is the adjoint of the variational equations for Eq. (2). Here f is a shorthand for the right-hand side (rhs) of Eq. (2); that is, $f = (E', W_f')$. Clearly the shape of the hodograph or velocity set

$$\nu(E) = [\,(E', W_f') \mid \eta,\ h \text{ admissible}\,]$$

is of paramount importance in the determination of the f that maximizes $\langle \lambda, f \rangle$. In general the velocity set depends on the energy E [because the rhs of (2) does] but not on the fuel used, since W_f is an "ignorable" coordinate.

Chattering Cruise—The Simplest Model

In order to provide some insights into the nature of the problem at hand, it is worthwhile to consider a much simplified model in which the functions $\bar{T}(h,M)$ and $\bar{Q}(h,M)$ are constant-valued and the drag coefficient is given by the usual parabolic form

$$C_D = C_{D0} + KC_L^2$$

with C_{D0} and K constant. Air density is taken constant. It is convenient to use scaled variables.

$$\hat{E} = E(L/D)_{\max} \quad \text{and} \quad \hat{W}_f = W_f V_{\text{md}} p / \bar{Q}$$

Here $(L/D)_{\max}$ refers to the usual maximum lift-to-drag ratio *at lift equals weight.* V_{md} is the speed for minimum drag, and p is the ratio of maximum thrust to minimum drag. Since the density has been assumed constant, there is no distinction between equivalent and true airspeeds. In terms of the scaled variables the system (2a) may be written

$$\hat{E}' = \eta p - \tfrac{1}{2}(u^2 + 1/u^2) \qquad W_f' = \eta p / u \tag{3}$$

where u is the nondimensional speed given by $u = V/V_{\text{md}}$. For this model and in terms of the scaled variables, the velocity set ν depends on E *only through the limits on altitude* (2b). For sufficiently large E, these bounds do not affect the minimum-fuel problem; thus they may be ignored. The set ν (shown in Fig. 1) depends on the particular aircraft through the parameter $p = T/D_{\min}$. To provide insight it is appropriate to give some details about the construction of the velocity set, shown in Fig. 1. For each fixed η the locus in $(\hat{E}', \hat{W}_f')$ is a distorted parabola with "symmetry" axis parallel to the $\hat{E}'$ coordinate axis. This becomes more apparent if we eliminate u from Eq. (3) to produce

$$\hat{E}' = \sigma - \tfrac{1}{2}[(\sigma/\hat{W}_f')^2 + (\hat{W}_f'/\sigma)]^2 = f(\hat{W}_f', \sigma) \tag{4}$$

where $\sigma = \eta p$. The boundary of the set ν consists of four parts, as indicated in Fig. 1. That portion labeled a is the locus from (3) with $\eta = 0$ and corresponds, of course, to gliding flight. The right-most extreme point arises with $u = 1$, that is, gliding flight at the speed for maximum lift-to-drag ratio. By virtue of our scaling, the coordinates are $(-1, 0)$. The portion labeled ζ is the envelope of the family Eq. 4 as σ varies from zero to p. From elementary analysis (see Ref. 11, pp. 456-461), points on the envelope must simultaneously satisfy, Eq. (4) and

$$f_\sigma = g(\hat{W}_f', \sigma) = 0 \tag{5}$$

The part of the boundary labeled c consists of the portion of the η-equals-one locus (maximum thrust) from the envelope to the loft ceiling. The final portion corresponds to flight at the loft ceiling for varying throttle settings. This part is not relevant to minimum-fuel flight and is sketched only roughly in Fig. 1. Finally, from the above discussion it should be clear that the family of velocity sets parameterized by p is nested; that is, if $p_1 < p_2$, then $\nu(p_1) \subset \nu(p_2)$.

For the optimal-control problem at hand, the significant feature of the set is that *it is not convex.* This is important because, in the absence of convexity, one cannot guarantee that an optimal control exists. Whereas, there is a greatest lower bound to the cost functional, it may be that no "smooth" control can produce that value.

To remedy the situation, it is necessary to extend the notion of an admissible control "function" to allow chattering controls.[9,16,17] In terms of Fig. 1 this means we add points to the hodograph figure to obtain the smallest convex set that contains the original set ν. This is the *convex hull* (CH) *of* ν [CH (ν)] and is shown in Fig. 2. The procedure is to apply the Maximum Principle with the set of allowable controls enlarged to generate CH(ν), rather than ν itself. As is evident from Fig. 2, the trajectory and cost generated using CH (ν) will be different if the λ function takes values in the cone between λ^1 and λ^2. Whether or not this occurs will depend on the specified values of $E(0)$ and $E(x_f)$.

For the case $E(0) = E(x_f)$ it is evident that controls which made E' equal to zero will satisfy the boundary conditions. Thus points along the W_f' axis in the hodograph figure are of interest. In Fig. 2, point B produces a lower value of fuel usage (lb/mile) than does point A, so in this case a "chattering" control is optimal. This "chattering" control can be

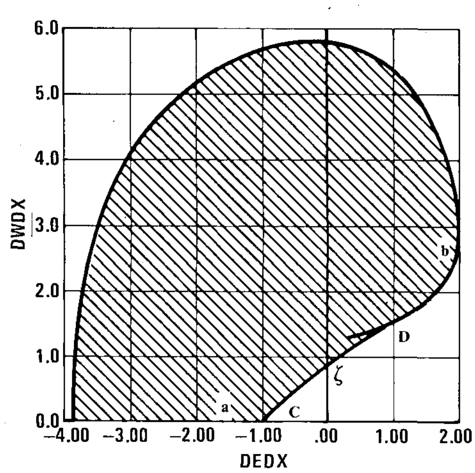

Fig. 1 Hodograph figure for simple model.

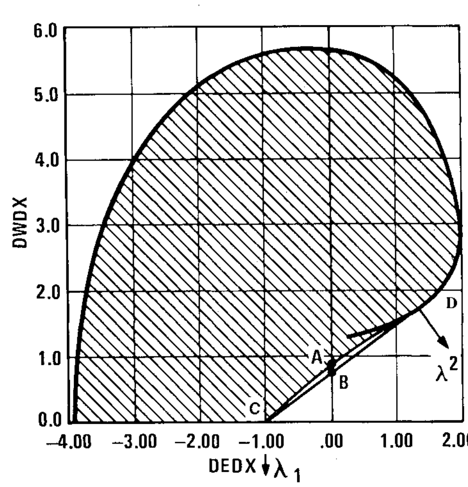

Fig. 2 Convex hull for velocity set in simple model.

approximated by a real control which switches between points C and D very often with time at each point apportioned so as to achieve $E' = 0$ on the average, i.e., point B in Fig. 2.

Perhaps the most significant measure of nonconvexity in the original velocity set ν is the ratio of the $\hat{W}_f'$ coordinates of the points A and B. Point A is on the envelope (2) where $\hat{E}'$ equals zero; it is the classical steady-cruise point. From Eqs. (4) and (5), with $\hat{E}' = 0$, one finds

$$\hat{W}_f / (\eta p) = (1/3)^{1/4}$$

while from the system (3), with $\hat{E}' = 0$, it follows that

$$u = 3^{1/4} \qquad \eta p = (2/3)^{1/2}$$

Substituting the ηp value yields

$$\hat{W}_{f_{ss}}' = (2/3)^{3/4} \doteq 0.8774$$

and converting back to the original dimensional variables gives

$$W_{f_{ss}}' = [(2/3)^{3/4}](D_{\min}/\bar{T})/(Q/V_{\rm md}) \tag{6}$$

where the subscript ss means steady state.

The point B in Fig. 2 is that point on the straight line joining C and D that has $\hat{E}'$ equal to zero. The coordinates of C are $(-1, 0)$. Point D is that point on the η-equals-one (maximum thrust) curve that produces a minimum value for the tangent of the angle θ shown in Fig. 2. It is readily seen that

$$\tan\theta = \hat{W}_f' / (1 + \hat{E}')$$

where $(\hat{E}', \hat{W}_f')$ are the coordinates of point D. Now $\hat{E}'$ can be expressed in terms of $\hat{W}f$ and σ from Eq. (4), and for all points D, σ equals p. Thus $\tan\theta$ is a function of $\hat{W}f$ and the parameter p. To maximize $\tan\theta$, one simply seeks a zero of the derivative with respect to $\hat{W}_f'$. This leads to the quadratic equation

$$(\hat{W}_f'/p)^4 + 2(p+1)(\hat{W}_f/p)^2 - 3 = 0$$

From this we obtain the $\hat{W}f$ coordinate of the point D

$$\hat{W}_f' = p\{[(p+1)^2 + 3]^{1/2} - (p+1)\}^{1/2} \tag{7}$$

and the corresponding $\hat{E}'$ coordinate can be found from Eq. (4) with η equal to one. Note that the required speed is found from the second of the equations (3),

$$u = \{[(p+1)^2 + 3]^{1/2} - (p+1)\}^{-1/2} \tag{8}$$

With points C and D in hand we can readily determine $\hat{W}_{f_{rss}}$ (rss stands for relaxed steady state), the $\hat{W}_f$ value at point B. Most simply we note from the definition of $\tan\theta$ that

$$\hat{W}_{f_{rss}}' = (\tan\theta)_{\min}$$

From Eqs. (3) and (4) we find

$$W_{f_{rss}}' = \frac{\bar{Q}[f(p+1)]^{3/2}}{2V_{\rm md}[(p+1)f(p+1) - 1]} \tag{9}$$

where $f(\beta) = [(\beta^2 + 3)^{1/2} - \beta]$.

Thus the ratio of the classical-cruise and chattering-cruise fuel-flow rates is

$$\frac{W_{f_{rss}}'}{W_{f_{ss}}'} = \left(\frac{3}{4}\right)^{3/4} \frac{p[f(p+1)]^{3/2}}{[(p+1)f(p+1) - 1]} \tag{10}$$

which depends only on the parameter $p = \bar{T}/D_{\min}$. The ratio is plotted in Fig. 3, from which it can be seen that the potential savings in fuel-flow rate increase with increasing p. Also shown in Fig. 3 is the nondimensional speed u, required for flight at point D. Recall that the condition B is approximated by "chattering" between points C (gliding flight at u equals one) and D [(maximum thrust at u from Eq. (8)].

Chattering Cruise—More Precise Models

Whereas the results of the preceding section provide insight into the benefits of chattering cruise, the underlying model must be made more accurate if one is to obtain quantitative results. In the most general model studied here, the functions $\bar{T}\ (h,M)$ and $\bar{Q}(h,M)$ are approximated via cubic-spline interpolation in tabular data. The drag-polar "parameters" $C_{DQ}\ (M)$ and $K(M)$ are similarly evaluated by cubic-spline approximation, as is the atmospheric density $\rho(h)$. The atmospheric temperature variation is computed from the usual tropospheric/stratospheric standard atmosphere formulas, while the speed of sound is found as $a = (\gamma RT)^{1/2}$.

For the general model the velocity sets $\nu(E)$ are, of course, energy dependent. To generate such a set for fixed E, we generate the fixed-throttle loci from Eq. (2) for a sequence of η values from zero to one. Each such locus is found by varying h from zero to the loft ceiling in sufficiently small steps. Typical results are illustrated in Figs. 4 and 5.

In addition to the sets $\nu(E)$ one also needs to generate the corresponding convex hulls. Specifically, one needs to determine a straight line from the right-most extreme point of the η-equals-zero locus that is tangent to the η-equals-one curve. This is the analog of (b') in Fig. 2. The procedure for doing this is to "march" along the η-equals-one curve from the loft ceiling limit to sea-level and, at each evaluation point to extrapolate the tangent line to the E' axis. By interpolation one can determine the altitude (and hence the point on the locus) that produces a tangent extension through the η-equals-zero extreme point. The intersection of the tangent line and the W_f' axis is the chattering point, and its W_f' coordinate is $W_{f_{rss}}'$.

Aircraft Studied

A total of three aircraft were studied. An F-4E model was fabricated using thrust and fuel-flow rate data from Headquarters USAF and aerodynamic data from Ref. 2. A turbojet powered RPV was modeled using thrust and aerodynamic data from NASA Ames. Fuel-flow rate was computed using a specific-fuel consumption of $0.6(1+M)$ h^{-1}.[2] Finally, a "Supercruiser" study configuration was used, with thrust and aerodynamic data again from NASA Ames. Fuel-flow was modeled as in the RPV case. Details of the tabulated data may be found in Ref. 5.

Numerical Results

At each energy one can, using the above procedures, determine $W_{f_{rss}}'$. The $W_{f_{ss}}'$ value is achieved by flying at a specific h, M, η combination, while the $W_{f_{rss}}'$ value requires

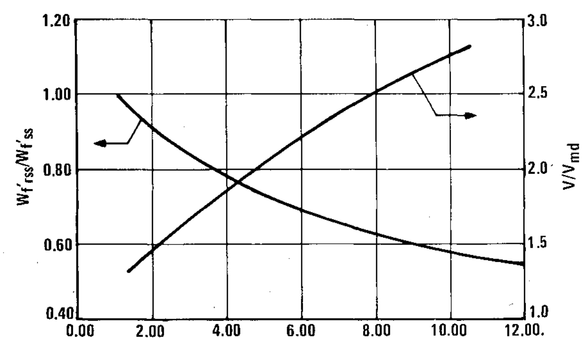

Fig. 3 $\hat{W}_{f_{rss}}/\hat{W}_{f_{ss}}$ vs $T_{\max}/D_{\min}$.

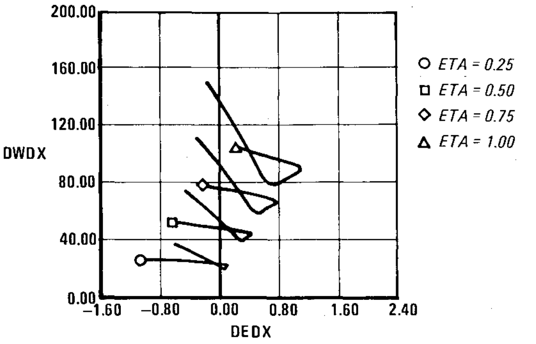

Fig. 4 Loci of constant throttle for Supercruiser, $E=20$.

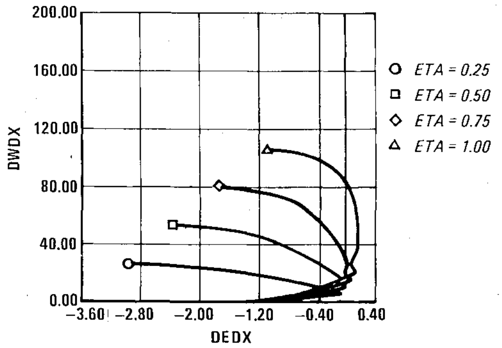

Fig. 5 Hodograph for Supercruiser, $E=60$.

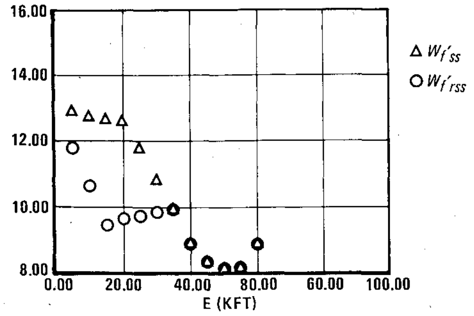

Fig. 6 Fuel savings for Supercruiser.

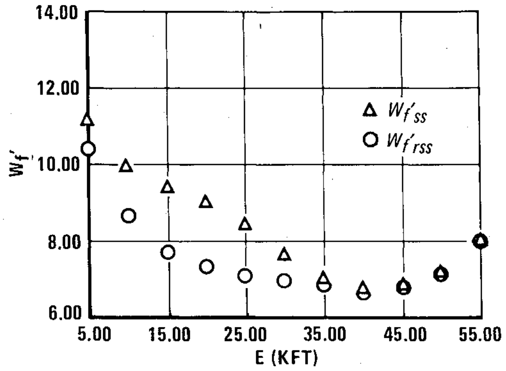

Fig. 7 Fuel savings for F-4E.

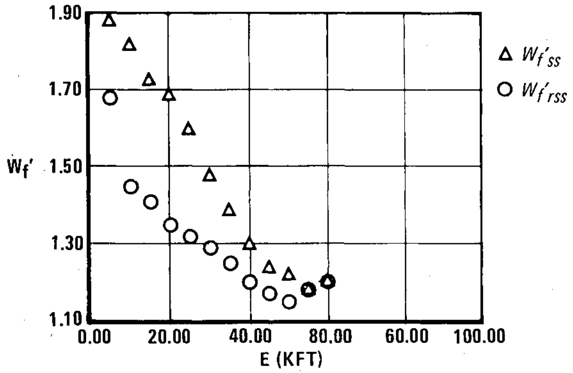

Fig. 8 Fuel savings for RPV.

chattering between a pair of such points, one at η equals zero and the other with η equal to one. Of primary significance is the global minimum found by selecting the energy level at which the minimum of $\{W'_{f_{ss}}(E), W'_{f_{rss}}(E)\}$ is attained. The energy that produces this minimum will be referred to as the *cruise energy* of the aircraft.

Shown in Fig. 6 are plots of $W'_{f_{ss}}$ and $W'_{f_{rss}}$ over a range of energy levels for the Supercruiser. Note that whereas chattering cruise shows a substantial improvement at low-energy levels [indicating nonconvexity of $\nu(E)$, as in Fig. 4], at higher-energy levels the improvement vanishes [indicating $\nu(E)$ is convex, as in Fig. 5]. Significantly, at the *cruise energy* for the Supercruiser (approximately 50,000 ft from Fig. 6) $\nu(E)$ is convex, so that the $W'_{f_{ss}}$ and $W'_{f_{rss}}$ values are identical. Figures 7 and 8 present the cruise results for the F-4E and the RPV, respectively. In these cases some small improvement remains at the cruise energy, with the RPV indicating about 5% better fuel mileage.

Recall that the chattering solution requires rapid switching between gliding flight at one (h,M) pair and full-throttle operation at a second (h,M) point. The required altitude "jumps" are indicated in Figs. 9-11. For the RPV the switches are between $h=41{,}000$ ft, $M=0.79$ in gliding flight and $h=37{,}000$ ft, $M=0.95$ with full throttle, at the cruise energy.

An intermediate case, with Mach dependence suppressed, was also examined. Results for the F-4E and the RPV are shown in Figs. 12 and 13. Note that, whereas the graphs of minimum-fuel usage vs energy are roughly the same as for the more complex case (Figs. 7 and 8), the behavior near the cruise energy is different, with the chattering solution showing more improvement than with Mach dependence.

Discussion

The simplest model studied indicates that potential fuel savings from chattering cruise depend only on the ratio of maximum thrust to minimum drag for the aircraft. The savings are appreciable, amounting to about 30% of classical-cruise fuel usage for the F-4E ($p \doteq 6$). These results agree with those reported by Gilbert and Parsons.[4] A more complex model featuring Mach-number and altitude dependence in the propulsion system and aerodynamic data predicts significant benefits at low energies, but not at high energies. At the cruise energy, which furnishes a global minimum of fuel usage in the variables h, η, and E, observed fuel savings are, at most, 5%.

Given these results, one naturally wonders: What became of the potential savings in the more complex case? Reviewing the Gilbert and Parsons[4] results, one notes for a model similar to the simplest one studied here, but including exponential decay with altitude of the functions $\bar{T}$, $\bar{Q}$, and the air density, that chattering cruise is never better than steady-state cruise at the cruise energy.

In Ref. 4 the problem is then modified by imposing an upper limit on altitude. This "forces" the solution to lower energies, where the benefits of chattering are more substantial (about 27% for the Supercruiser at 15,000 ft). Unfortunately, there does not seem to be any convincing reason for imposing such a limit operationally.

To provide some insight into the decrease in fuel-saving realized when the model is made more complex, it is instructive to consider some additional geometry in the velocity sets $\nu(E)$. Note that because of the assumption that fuel-flow rate varies linearly with thrust, the loci of constant-altitude points (varying η) in the hodograph are straight lines. The

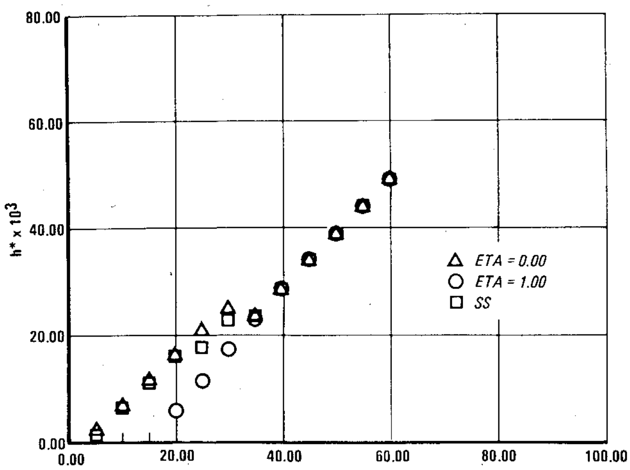

Fig. 9 Altitudes for chattering cruise, Supercruiser.

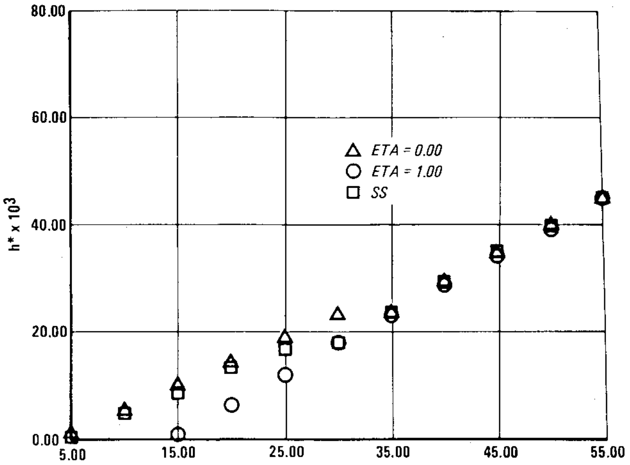

Fig. 10 Altitudes for chattering cruise, F-4E.

slope of any such line is, from Eq. (3),

$$s(h,E)=\frac{\mathrm{d}W_f'}{\mathrm{d}E'}=\frac{\bar{Q}(h,M)\,W}{\bar{T}(h,M)\,V} \tag{11}$$

where M and V are functions of h and E. Note that the ratio $\bar{Q}/\bar{T}$ is the specific fuel consumption, say $c(h,M)$. Consider now the right-most extreme point on the η-equals-zero curve, which corresponds to minimum-drag flight at altitude $\hat{h}$. A constant-altitude locus, say L, emanates from it to some point on the maximum-thrust curve. If the constant-altitude loci (straight lines) emanating from points near the $\hat{h}$ point on the η-equals-zero locus cross L, then the set $\nu(E)$ is not convex. Whether or not such crossings occur is a somewhat difficult issue, complicated by the fact that one is only interested in crossings that occur for η values between zero and one.

In order to determine the existence of such crossings, consider a family of constant-altitude lines from points on the η-equal-zero locus as the altitude varies from the minimum-drag value. Since $\hat{h}$ is the minimum-drag point, the "motion" of the point on the η-equals-zero curve as h varies from $\hat{h}$ is second order in $(h-\hat{h})$. ($\mathrm{d}E'/\mathrm{d}h=0$ at $\hat{h}$, where the derivative is taken at constant energy.) Thus, if the slope, $s(h,E)$ in Eq. (11), can be made smaller than $s(\hat{h},E)$, then crossings do occur, and the set ν (E) is nonconvex. This would seem to be the generic case (note that altitude can be varied on either side of $\hat{h}$), and would fail to happen only if

$$\frac{\partial s}{\partial h}=\frac{cW}{V}\left(\frac{1}{c}\frac{\partial c}{\partial h}-\frac{1}{V}\frac{\partial V}{\partial h}\right) \tag{12}$$

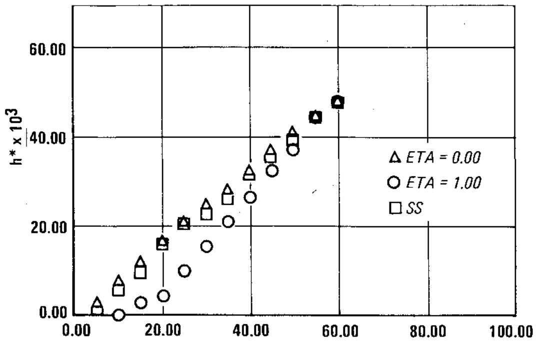

Fig. 11 Altitudes for chattering cruise, RPV.

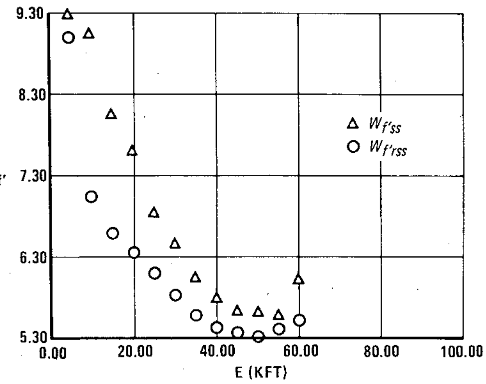

Fig. 12 Fuel savings for F-4E, $M=0$.

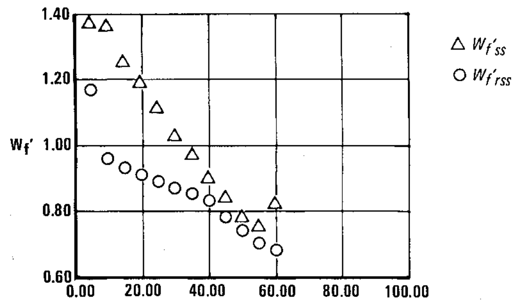

Fig. 13 Fuel savings for RPV, $M=0$.

were zero. In Eq. (12) the derivatives are taken at constant energy. The above argument is, of course, completely local and only indicates nonconvexity in the neighborhood of the h-equals-$\hat{h}$ point on the η-equals-zero locus, that is, points like C in Fig. 1.

At this point an unexpected connection with the work of Speyer[12] appears, in which an intermediate model *using thrust and path angle as controls* was studied. In Ref. 12 Speyer shows that, if the quantity on the rhs of Eq. (12) is nonzero, then cruise arcs are not minimizing. It is very interesting to note that this *same* condition guarantees at least local nonconvexity in the hodograph of the *energy model*.

In the complex models, as energy increases, the parts of the boundary of $\nu(E)$ determined by the envelope and the maximum-thrust-locus (labeled B and C, respectively, in Fig. 1) move downward and to the left. Owing to the leftward shift, the chattering point B (in Fig. 2) approaches the maximum-thrust point D, and the benefits of chattering are diminished. Thus, the above argument, that the envelope is not convex near the gliding-flight point C, cannot be invoked to establish nonconvexity near B. In fact, for the analytical model considered by Gilbert and Parsons,[4] this "down-and-left" shift of the boundary continues as E increases until the maximum E' value on the maximum-thrust curve is zero. In this case points B and D coincide, and nonconvexity of $\nu(E)$ near C is not relevant.

The results from the study with no Mach dependence indicate that such dependence can have a marked effect on the amount of improvement, if any, produced by chattering. Note that in the RPV case (Fig. 13), with Mach dependence suppressed, the specific fuel consumption is constant-valued, as in the analytical models used here and in Ref. 4. In fact, for the RPV with Mach dependence suppressed, the mathematical model here differs from that of the Gilbert and Parsons analysis[2] only in the altitude functions used to describe the maximum thrust and the air density. This suggests that the form of such altitude dependence can have a significant effect in chattering-cruise studies.

Concluding Remarks

The energy-approximation studies with various system models show the expected relationship between nonconvexity of the hodograph figure, or velocity set, and the sometime superiority of chattering cruise over classical cruise. Modeling details make a day-and-night difference, however. For example, the simplified model omitting both Mach-number and altitude dependence has substantial superiority of chattering over classical steady cruise as a standard feature, while the Ref. 4 modeling of thrust and density vs altitude with the same scale height predicts no chattering-cruise superiority unless a rather unlikely altitude constraint is introduced. Mach-number effects on aerodynamic and propulsion-system data seem to have an important effect.

The practical implication of the study is that chattering cruise offers at best only a small advantage over conventional cruise. It is important to note, however, that these results apply to aircraft that were "conventionally" designed. In particular, the thrust characteristics of the powerplant are, in some sense, "tailored" to the aerodynamic characteristics of the airframe. It may well be that given some design freedom one could demonstrate a chattering cruise that offers significant fuel savings over conventional cruise.

On a more theoretical level, it is hoped that the present study contributes to an understanding of some modeling problems in flight mechanics. Two points seem worthy of emphasis:

1) The condition that guarantees nonconvexity in the hodograph figure of the energy model is identical to that which rules out partial-throttle cruise in an intermediate model (path angle as a control).

2) While the energy model hodograph is generically nonconvex, the lack of convexity does not imply chattering cruise. In energy models it frequently happens that at the best energy for cruise, the hodograph touches the W_f' axis (E' equals zero) in only one point, thus points B and C (see Fig. 1) coincide and the chattering degenerates to full-throttle cruise.

The present study has a weakness in common with others in the literature; viz., no account has been taken of time constraints, essential to serious study of supersonic cruise, for example. The dimension of the hodograph space is 3 instead of 2 for studies with time constraints, which complicates the analysis considerably, and it is therefore not surprising that concentration to date has been on the time-free case.

No attempt at study of oscillatory cruise with higher-order models has been made in the presently reported work. There are obvious difficulties in using a reduced-order chattering solution as a reference (or "outer") solution for some type of expansion; however, explorations along such directions would seem to be of future interest. It should be noted that oscillations in a point-mass model need not be related to chattering in the energy model. Several recent studies[1,14] suggest that kinetic/potential energy interchange can produce fuel savings over conventional cruise.

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